

COSMOLOGICAL/BLACK-HOLE UNIFIED THEORY AS A CONSTRAINED SYSTEM

J. A. Nieto^{a1} and G. Avila^{b2}

*^aFacultad de Ciencias Físico-Matemáticas de la Universidad Autónoma
de Sinaloa, C.P. 80010, Culiacán Sinaloa, México*

*^bDepartamento de Investigación en Física de la Universidad
de Sonora, C.P. 83000, Hermosillo, Sonora, México*

Abstract

Using a Lagrangian formalism we unify the cosmology and black-hole concepts. Specifically, we identify these two physical scenarios as part of a 2-dimensional metric, which arises from a Lagrangian (with constraints) derived from the Einstein-Hilbert action. In particular, we show that the Friedman-Robertson-Walker cosmological model and the Schwarzschild black-hole solution are both a consequence of a such Lagrangian.

Key words; Cosmological models, Kaluza-Klein theory, Black-Holes.

Pacs No.: 98.80.Dr, 12.10.Gq, 04.50.+h.

Agust 2014

¹nieto@uas.uasnet.mx

²gavilasierra@posgrado.cifus.uson.mx

Recently, it was shown [1]-[2] (see also Ref. [3]) that both the Friedman-Robertson-Walker cosmological model (cosmology) or the Schwarchild black-hole solution (black-hole) can be derived from a first order Lagrangian associated with a constrained system. By considering a particular ansatz such a Lagrangian is obtained from the Einstein-Hilbert action. In this work we generalize such a formalism showing that the cosmology and black-hole concepts can be obtained as a limit case of our more general constrained Lagrangian formalism.

Our approach may be physically interesting for a number of reasons. First, one may use the complete mathematical tools of Lagrangians for constrained systems in order to study a number of symmetries underlying the cosmology and black-hole solutions. Second, a unified treatment of cosmology and black-holes may be of particular interest in the context of string theory [4] or M-theory [5]. One reason for this it is because M-theory predicts among other things that our universe can be described by a brane-world [6]. So, it appears attractive to search for a brane-world/black-hole correspondence from the point of view of M-theory.

It is important to mention that the idea of combining the cosmology and black-hole concepts has a long history (see Ref. [7] and references therein). Let just mention that perhaps one of the early ideas is the so-called Swiss cheese universe which is constructed by cutting out spheres from cosmology universe and collapsing the matter down into black holes [8]. At present, the subject is known as cosmological black holes (or Schwarzschild cosmology, or black-hole cosmological model (see Ref. [7] and references therein)). In any case, the main idea on these approaches is to combine the Hubble radius r_H of the observable universe with its Schwarzschild radius r_S of a black-hole [9]-[10]. The main difference between our formalism and such previous works is that in our approach both cosmology and black-hole are unified through a dynamic two dimensional metric without requiring any combination between radius r_H and r_S .

Consider the Hilbert-Einstein action in $n + D + d$ -dimensions,

$$S = \frac{1}{V_{D+d}} \int d^{n+D+d}x \sqrt{-g} (R - 2\Lambda), \quad (1)$$

and assume the ansatz

$$ds^2 = g_{AB}(x^C) dx^A dx^B + a^2(x^C) d^D\Omega + b^2(x^C) d^d\Sigma, \quad (2)$$

where the indices A, B run from 1 to n . Here, the expressions $d^D\Omega \equiv \tilde{g}_{ab}(x^d) dx^a dx^b$ and $d^d\Sigma \equiv \hat{g}_{ij}(x^k) dx^i dx^j$ correspond to a D -dimensional and d -dimensional ho-

mogenous spatial spaces, with constant curvature $k_1 = 0, \pm 1$ and $k_2 = 0, \pm 1$, respectively.

It can be shown that using (2) the action (1) is reduced to

$$\begin{aligned}
S = \int d^n x \sqrt{g} a^D b^d \{ & -2D a^{-1} \mathcal{D}_A \partial^A a - D(D-1) g^{AB} a^{-2} \partial_A a \partial_B a \\
& -2d b^{-1} \mathcal{D}_A \partial^A b - d(d-1) g^{AB} b^{-2} \partial_A b \partial_B b \\
& -2Dd(b^{-1} a^{-1}) g^{AB} \partial_A a \partial_B b + \bar{R} + a^{-2} \tilde{R} + b^{-2} \hat{R} - 2\Lambda \}.
\end{aligned} \tag{3}$$

which can be rewritten as

$$\begin{aligned}
S = - \int d^n x \sqrt{g} \{ & \mathcal{D}_A (2D a^{D-1} \partial^A a b^d + 2d b^{d-1} \partial^A b a^D) \\
& -D(D-1) g^{AB} a^{D-2} b^d \partial_A a \partial_B a - d(d-1) g^{AB} a^D b^{d-2} \partial_A b \partial_B b \\
& -2Dd(a^{D-1} b^{d-1}) g^{AB} \partial_A a \partial_B b - a^D b^d (\bar{R} + a^{-2} \tilde{R} + b^{-2} \hat{R} - 2\Lambda) \},
\end{aligned} \tag{4}$$

where \mathcal{D}_A is a covariant derivative associated with g_{AB} . Dropping the total derivative in (4) one obtains [1],

$$\begin{aligned}
S = \int d^n x \sqrt{g} a^D b^d \{ & D(D-1) g^{AB} a^{-2} \partial_A a \partial_B a + d(d-1) g^{AB} b^{-2} \partial_A b \partial_B b \\
& +2Dd(a^{-1} b^{-1}) g^{AB} \partial_A a \partial_B b + (\bar{R} + a^{-2} \tilde{R} + b^{-2} \hat{R} - 2\Lambda) \}.
\end{aligned} \tag{5}$$

Here, \bar{R} , \tilde{R} and \hat{R} are the curvature scalars associated with $g_{AB}(x^C)$, $\tilde{g}_{ab}(x^d)$ and $\hat{g}_{ij}(x^k)$, respectively (see Ref. [1] and [2] for details).

It is worth mentioning that provided $n \neq 2$ and $\bar{R} + a^{-2} \tilde{R} + b^{-2} \hat{R} - 2\Lambda = 0$, one finds that the action (5) is invariant under the duality transformation [11],

$$\begin{aligned}
a & \rightarrow \frac{1}{a}, \\
b & \rightarrow \frac{1}{b}, \\
g_{AB} & \rightarrow a^{\frac{4D}{n-2}} b^{\frac{4d}{n-2}} g_{AB}.
\end{aligned} \tag{6}$$

(see also Refs. [1] and [12]). This means that the case $n = 2$ is distinguished from any other n value. Therefore, from point of view of the duality transformation (6), the model (5) with 2-dimensional metric $g_{AB}(x^C)$

is an exceptional case. In some sense the duality symmetry is playing here the analogue role as the Weyl invariance in p -brane physics (see Ref. [4] and references therein), which implies that the 1-brane (string) theory is an exceptional case.

A particular case of (2) is expressed by the line element

$$ds^2 = -N^2(t)dt^2 + a^2(t)d^D\Omega + b^2(t)d^d\Sigma, \quad (7)$$

In this case the Einstein-Hilbert action (5) is simplified to

$$\begin{aligned} S = \int dt \{ & N^{-1}a^Db^d[D(D-1)a^{-2}\dot{a}^2 + d(d-1)b^{-2}\dot{b}^2 + 2dDa^{-1}\dot{a}b^{-1}\dot{b}] \\ & -D(D-1)k_1Na^{D-2}b^d - d(d-1)k_2Na^Db^{d-2} + 2\Lambda Na^Db^d\}. \end{aligned} \quad (8)$$

One can show that the field equations for a cosmology model in $1 + D + d$ follows from (8) [1].

The problem with (7) is that it can be obtained from (2) by taking $n = 2$ and choosing $g_{11}(x^C) = -N^2$, $g_{12}(x^C) = g_{21}(x^C) = 0$ and $g_{22}(x^C) = 0$. This means that the 2-dimensional metric $g_{AB}(x^C)$ in (5) is singular, in the sense that $\det g_{AB}(x^C) = 0$. So, the question arises starting from (5) and assuming $\det g_{AB}(x^C) \neq 0$ how can one obtains the action (8)? The answer to this question may be solved by performing a different kind of projection. Let us rewrite the ansatz (2) in the form

$$ds^2 = g_{AB}(x^1, x^2)dx^A dx^B + a^2(x^1, x^2)d^D\Omega + b^2(x^1, x^2)d^d\Sigma. \quad (9)$$

Let us assume that $g_{12}(x^1, x^2) = 0$. This leads to

$$ds^2 = g_{11}(x^1, x^2)dx^1 dx^1 + g_{22}(x^1, x^2)dx^2 dx^2 + a^2(x^1, x^2)d^D\Omega + b^2(x^1, x^2)d^d\Sigma. \quad (10)$$

By performing the projections $g_{11}(x^1, x^2) \longrightarrow g_{11}(x^1) = -N^2$, $g_{22}(x^1, x^2) \longrightarrow g_{22}(x^1) = a^2(x^1)$ and $b^2(x^1, x^2) \longrightarrow b^2(x^1)$ the line element (10) becomes

$$ds^2 = -N^2(x^1)dx^1 dx^1 + a^2(x^1)d^{D+1}\Omega + b^2(x^1)d^d\Sigma. \quad (11)$$

Here, we have defined $d^{D+1}\Omega = dx^2 dx^2 + d^D\Omega$. Therefore, one has discovered that (11) has exactly the same form as (7) and therefore the action (8) follows, provided one makes the extension

$$g_{ab} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & g_{ab} \end{pmatrix}. \quad (12)$$

In the case of black-holes one can make the reductions $g_{11}(x^1, x^2) \longrightarrow g_{11}(x^2) = -e^{f(r)}$ and $g_{22}(x^1, x^2) \longrightarrow g_{22}(x^2) = e^{h(r)}$, $a^2(x^1, x^2) \longrightarrow \varphi^2(r)$ and also $b^2(x^1, x^2) = b^2(r)$, with $x^2 = r$. One finds that the line element (10) becomes

$$ds^2 = -e^{f(r)}(dx^1)^2 + e^{h(r)}dr^2 + \varphi^2(r)d^D\Omega + b^2(r)d^d\Sigma. \quad (13)$$

In this case, one finds that the curvature scalar R becomes

$$\begin{aligned} R = e^{-h} \{ & -\ddot{f} - \frac{\dot{f}^2}{2} + \frac{\dot{f}\dot{h}}{2} + D(\dot{h} - \dot{f})\frac{\dot{\varphi}}{\varphi} - 2D\frac{\ddot{\varphi}}{\varphi} \\ & - D(D-1)\frac{\dot{\varphi}^2}{\varphi^2} + d(\dot{h} - \dot{f})\frac{\dot{b}}{b} - 2d\frac{\ddot{b}}{b} - d(d-1)\frac{\dot{b}^2}{b^2} - 2Dd\frac{\dot{\varphi}\dot{b}}{\varphi b} \} \\ & + k_1 D(D-1)\varphi^{-2} + k_2 d(d-1)b^{-2}. \end{aligned} \quad (14)$$

Here, the dot in any quantity \dot{A} means derivative with respect r . On the other hand, we have

$$\sqrt{-\bar{g}} = e^{\frac{f+h}{2}} \varphi^D b^d \sqrt{\tilde{g}} \sqrt{\hat{g}}, \quad (15)$$

where \bar{g} , \tilde{g} and \hat{g} denote the determinant of $g_{\mu\nu}$, \tilde{g}_{ab} and \hat{g}_{ij} , respectively. Consequently, one can show that the higher dimensional Einstein-Hilbert action

$$S = \frac{1}{V_{D+d}} \int_{M^{2+D+d}} \sqrt{-\bar{g}} (R - 2\Lambda), \quad (16)$$

can be simplified to the form

$$\begin{aligned} S = \int dr \{ & (e^{\frac{f-h}{2}} \varphi^D b^d) [-\ddot{f} - \frac{\dot{f}^2}{2} + \frac{\dot{f}\dot{h}}{2} + D(\dot{h} - \dot{f})\frac{\dot{\varphi}}{\varphi} - 2D\frac{\ddot{\varphi}}{\varphi} \\ & - D(D-1)\frac{\dot{\varphi}^2}{\varphi^2} + d(\dot{h} - \dot{f})\frac{\dot{b}}{b} - 2d\frac{\ddot{b}}{b} - d(d-1)\frac{\dot{b}^2}{b^2} - 2Dd\frac{\dot{\varphi}\dot{b}}{\varphi b}] \\ & + k_1 D(D-1)e^{\frac{f+h}{2}} \varphi^{D-2} b^d + k_2 d(d-1)e^{\frac{f+h}{2}} \varphi^D b^{d-2} - 2e^{\frac{f+h}{2}} \varphi^D b^d \Lambda \}. \end{aligned} \quad (17)$$

Furthermore, one can prove that (17) can be rewritten as

$$\begin{aligned} S = \int dr \{ & -2\frac{d}{dr}(e^{\frac{-h}{2}} \frac{d}{dr}(\varphi^D b^d e^{\frac{f}{2}})) + (e^{\frac{f-h}{2}} \varphi^D b^d) [D(D-1)\frac{\dot{\varphi}^2}{\varphi^2} + d(d-1)\frac{\dot{b}^2}{b^2} \\ & + \frac{D\dot{\varphi}\dot{f}}{\varphi} + \frac{d\dot{b}\dot{f}}{b} + 2Dd\frac{\dot{b}\dot{\varphi}}{b\varphi}] \\ & + k_1 D(D-1)e^{\frac{f+h}{2}} \varphi^{D-2} b^d + k_2 d(d-1)e^{\frac{f+h}{2}} \varphi^D b^{d-2} - 2e^{\frac{f+h}{2}} \varphi^D b^d \Lambda \}. \end{aligned} \quad (18)$$

So, up to total derivative, the action (18) is reduced to

$$\begin{aligned}
S = \int dr \{ & (\Omega^{-1} \mathcal{F} \varphi^D b^d) [D(D-1) \frac{\dot{\varphi}^2}{\varphi^2} + d(d-1) \frac{\dot{b}^2}{b^2} + 2D \frac{\dot{\mathcal{F}}}{\mathcal{F}} \frac{\dot{\varphi}}{\varphi} + 2d \frac{\dot{\mathcal{F}}}{\mathcal{F}} \frac{\dot{b}}{b} + 2D d \frac{\dot{b}}{b} \frac{\dot{\varphi}}{\varphi}] \\
& + \Omega \mathcal{F} [k_1 D(D-1) \varphi^{D-2} b^d + k_2 d(d-1) \varphi^D b^{d-2} - 2\varphi^D b^d \Lambda] \}.
\end{aligned} \tag{19}$$

Here, we used the notation $\mathcal{F} \equiv e^{\frac{f}{2}}$ and $\Omega \equiv e^{\frac{h}{2}}$. The important thing is that by setting $n = 2$ (19) can be obtained in straightforward way from (5). From the action (19) one can obtain the field equations whose solution, in the case $b = 0$, corresponds to the the well known higher dimensional Schwarchild black-hole solution, namely

$$ds^2 = -\left(1 - \frac{k}{\varphi^{D-1}(r)}\right)(dx^1)^2 + \frac{d\varphi^2}{\left(1 - \frac{k}{\varphi^{D-1}(r)}\right)} + \varphi^2(r) d^D \Omega. \tag{20}$$

Summarizing, in this article by emphasizing that from the point of view of the duality symmetry $a \rightarrow \frac{1}{a}$ and $b \rightarrow \frac{1}{b}$ the case $n = 2$ seems to be exceptional. We showed that the Einstein-Hilbert action in $n + D + d$ -dimensions is reduced to the action (5) which for the case of $n = 2$ contains the unified dynamics of both cosmology and black-holes.

References

- [1] J. A. Nieto, M. P. Ryan, O. Velarde, C. M. Yee, *Int.J.Mod.Phys. A***19** (2004) 2131, hep-th/0401145.
- [2] J. A. Nieto, E. A. Leon, V. M. Villamueva, *Int. J. Mod. Phys. D* **22** (2013) 1350047; arXiv:1302.1469 [gr-qc]
- [3] E. A. Leon, R. Nunez-Lopez, A. Lipovka, J. A. Nieto, *Mod. Phys. Lett. A* **26** (2011) 805; arXiv:1012.3556 [gr-qc]
- [4] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory Vol. I and II* (Cambridge, Uk: Univ. Pr. (1987)).
- [5] E. Martinec, *Geometrical Structure of M Theory*; hep-th/9608017.
- [6] P. Brax and C. van de Bruck, *Class. Quant. Grav.* **20** (2003) R201; hep-th/0303095.
- [7] M. L. McClure, "Cosmological Black Holes as Models of Cosmological Inhomogeneities", PhD. Thesis, Department of Astronomy and Astrophysics, Toronto University 2006.
- [8] A. Einstein and E. G. Straus, *Rev. Mod. Phys.*,**17** (1945) 120.
- [9] R. K. Pathria, "The Universe as a Black Hole". *Nature* **240** (5379) (1972) 298.
- [10] I. J. Good, "Chinese universes". *Physics Today* **25** (7) (1972) 15.
- [11] V. I. Tkach, J. Socorro, J. J. Rosales and J. A. Nieto, *Phys. Rev.* **D60**, 067503 (1999); hep-th/9807129.
- [12] M. Gasperini; *Elementary Introduction to Pre-Big-Bang Cosmology and to the Relic Graviton Background*, proceedings of SIGRAV Graduate School in Contemporary Relativity and Gravitational Physics, Villa Olmo, Como, Italy, 19-24 Apr (1999); hep-th/9907067.